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Tile Digit Sets of Integral Self-affine Tilings

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Abstract: Integral self-affine tilings generated by an expanding integer matrix $\mathbb{A} \in M_n(\mathbb{Z})$ and $\mathscr{D} = \{d_1, d_2, \cdots, d_m\} \subseteq \mathbb{Z}^n$ have been studied by many works. An important problem is to decide when a digit set gives us a tile (and then we call it a tile digit set). It is shown that the standard digit sets by Bandt, product form digit sets by Lagarias and Wang, and weak-product form digit sets in $\mathbb{R}^1$ by Lau and Rao are tile digit sets. In this paper, we generalize the notion of weak product form to higher dimensions and prove that they are tile digit sets.

Keywords: integral self-affine tiling; standard digit set; product form digit set; weak-product form digit set

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1 Introduction

Let $\mathbb{A}$ be an expanding integer matrix in $M_n(\mathbb{Z})$ (all eigenvalues $\lambda_i(\mathbb{A}) > 1$). Denote $m = |\det(\mathbb{A})|$. Let $\mathscr{D} = \{d_1, d_2, \cdots, d_m\} \subseteq \mathbb{Z}^n$ be a finite set of vectors, called digit set. Then $\phi_i(x) = \mathbb{A}^{-1}(x + d_i)$, $1 \le i \le m$ are all contractions[1]. There is a unique compact set $T = T(\mathbb{A}, \mathscr{D})$ satisfying[2]

$$T = \bigcup_{i=1}^{m} \phi_i(T). \tag{1}$$

T is given explicitly by $T := \big\{\sum\limits_{k=1}^{\infty} \mathbb{A}^{-k} d_k : d_k \in \mathscr{D}\big\}$. An equivalent form of the functional equation (1) is $\mathbb{A}(T) = \bigcup\limits_{i=1}^{m} (T + d_i)$. When the attractor $T(\mathbb{A}, \mathscr{D})$ has a positive Lebesgue measure, we call it an integral self-affine tile, and $\mathscr{D}$ a tile digit set. In this case, it is well known that such T tiles $\mathbb{R}^n$ by some translation set $\mathscr{S} \subseteq \mathbb{Z}^n$ (see [1]).

First, let us introduce some notations. $\mathscr{D}_{\mathbb{A},k} = \big\{\sum\limits_{j=0}^{k-1} \mathbb{A}^j d_{i_j} : \text{all } d_{i_j} \in \mathscr{D}\big\}$, $\mathscr{D}_{\mathbb{A},\infty} = \bigcup\limits_{k=1}^{\infty} \mathscr{D}_{\mathbb{A},k}$. Note that $0 \in \mathscr{D}$ implies that $\mathscr{D}_{\mathbb{A},k} \subseteq \mathscr{D}_{\mathbb{A},k+1}$ for all $k \ge 1$. We say that a set $\mathcal{V} \subseteq \mathbb{R}^n$ is uniformly discrete if there exists $\delta > 0$ such that $v, v' \in \mathcal{V}$ implies $|v - v'| > \delta$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.

The following theorem gives criterion for $T(\mathbb{A}, \mathscr{D})$ being a self-affine tile[1].

Theorem 1.1 (Interior Theorem) Let $\mathbb{A} \in M_n(\mathbb{R})$ be an expanding matrix such that $|\det(\mathbb{A})| = m$ is an integer. Let $\mathscr{D} \subseteq \mathbb{R}^n$ have cardinality m, and suppose that $0 \in \mathscr{D}$. The following four conditions are equivalent.

(i) $T(\mathbb{A}, \mathscr{D})$ has positive Lebesgue measure;

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(ii) $T(\mathbb{A},\mathscr{D})$ has nonempty interior;

(iii) $T(\mathbb{A},\mathscr{D})$ is the closure of its interior T°, and its boundary $\partial T := T - T^{\circ}$ has Lebesgue measure zero;

(iv) For each $k \geq 1$, $\mathscr{D}_{\mathbb{A},k}$ has m^k distinct elements and $\mathscr{D}_{\mathbb{A},\infty}$ is a uniformly discrete set.

If $\mathscr{D}$ is a standard digit set, then it is a tile digit set[3]. It is introduced in [4] digit sets of product form and showed they are tile digit set. In the 1-dimension case, it is defined in [5] a class of weak product forms and showed that they are tile digit sets. In this paper, we generalize the result of [5] to higher dimensions.

2 Weak product-form digit sets

Recall that a digit set $(\mathbb{A},\mathscr{D})$ is a product-form digit set if $\mathscr{D}$ has an additive factorization

$$\mathscr{D} = \mathbb{A}^{f(1)}(\mathscr{E}_1) + \mathbb{A}^{f(2)}(\mathscr{E}_2) + \cdots + \mathbb{A}^{f(r)}(\mathscr{E}_r),$$

in which $r \geq 2$, and $0 \leq f(1) \leq f(2) \leq \cdots \leq f(r)$, and where $\mathscr{E}_1, \mathscr{E}_2, \cdots, \mathscr{E}_r \subseteq \mathbb{Z}^n$, each have $0 \in \mathscr{E}_i$, and $\mathscr{E} := \mathscr{E}_1 + \mathscr{E}_2 + \cdots + \mathscr{E}_r$ is a complete set of coset representatives of $\mathbb{Z}^n/\mathbb{A}(\mathbb{Z}^n)$, that is $\sharp(\mathscr{E}) = \sharp(\mathscr{E}_1)\sharp(\mathscr{E}_2)\cdots\sharp(\mathscr{E}_r) = |\det(\mathbb{A})|$. If some $f(i) > 0$, then this is a nonstandard digit set and we have the following theorem[4].

Theorem 2.1 A product-form digit set tile $T(\mathbb{A},\mathscr{D})$ is a measure-disjoint union of translates of $T(\mathbb{A},\mathscr{E})$, and has measure

$$\mu\big(T(\mathbb{A},\mathscr{D})\big) = \mu\big(T(\mathbb{A},\mathscr{E})\big)\prod_{i=1}^{r}\sharp(\mathscr{E}_i)^{f(i)}. \tag{2}$$

It is showed in [4] that, for the above $\mathscr{D}$ and $\mathscr{E}$, there exists $\mathcal{W} \subseteq \mathbb{Z}^n$ such that

$$\mathbb{A}^{f(r)}T(\mathbb{A},\mathscr{E}) = T(\mathbb{A},\mathscr{D}) + \mathcal{W}. \tag{3}$$

Since $(T(\mathbb{A},\mathscr{E}),\mathbb{Z}^n)$ is a tiling of $\mathbb{R}^n$, it follows that $(\mathbb{A}^{f(r)}T(\mathbb{A},\mathscr{E}),\mathbb{A}^{f(r)}\mathbb{Z}^n)$ is also a tiling of $\mathbb{R}^n$. Therefore $(T(\mathbb{A},\mathscr{D}),\mathcal{W}+\mathbb{A}^{f(r)}\mathbb{Z}^n)$ is a tiling of $\mathbb{R}^n$. We denote that

$$\mathcal{J} = \mathcal{W} + \mathbb{A}^{f(r)}\mathbb{Z}^n. \tag{4}$$

We need the following simple result in [3].

Lemma 2.1[6] If $\mathbb{A}^p T(\mathbb{A},\mathscr{D})$ can be tiled by $T(\mathbb{A},\mathscr{D})$ with two translation sets $\mathcal{J}$ and $\mathcal{J}'$ where $p \in \mathbb{Z}$, then $\mathcal{J} = \mathcal{J}'$.

Theorem 2.2 Let $\mathscr{D} \subseteq \mathbb{Z}^n$ be a product-form digit set, and $\mathcal{J} = \mathcal{W} + \mathbb{A}^{f(r)}\mathbb{Z}^n$. Then $\mathcal{J} = \mathbb{A}\mathcal{J} + \mathscr{D}$.

Proof Since

$$\mathbb{A}T(\mathbb{A},\mathscr{E}) = \mathbb{A}\Big\{\sum_{k=1}^{\infty}\mathbb{A}^{-k}e_k : e_k \in \mathscr{E}\Big\} = \Big\{\sum_{k=0}^{\infty}\mathbb{A}^{-k}e_k : e_k \in \mathscr{E}\Big\}$$

$$= \Big\{e_0 + \sum_{k=1}^{\infty}\mathbb{A}^{-k}e_k : e_0, e_k \in \mathscr{E}\Big\} = \mathscr{E} + T(\mathbb{A},\mathscr{E}),$$

$$\mathbb{A}^{f(r)+1}T(\mathbb{A},\mathscr{E}) = \mathbb{A}^{f(r)}\big(\mathscr{E} + T(\mathbb{A},\mathscr{E})\big) = \mathbb{A}^{f(r)}\mathscr{E} + \mathcal{W} + T(\mathbb{A},\mathscr{D}).$$

And

$$\mathbb{A}^{f(r)+1}T(\mathbb{A},\mathscr{E}) = \mathbb{A}\big(\mathcal{W}+T(\mathbb{A},\mathscr{D})\big) = \mathbb{A}\mathcal{W}+\mathscr{D}+T(\mathbb{A},\mathscr{D}).$$

Then by Lemma 2.1, $\mathbb{A}^{f(r)}\mathscr{E}+\mathcal{W} = \mathbb{A}\mathcal{W}+\mathscr{D}$. Hence

$$\begin{aligned}\mathbb{A}\mathcal{J}+\mathscr{D} &= \mathbb{A}\mathcal{W}+\mathbb{A}^{f(r)+1}\mathbb{Z}^n+\mathscr{D} = \mathbb{A}^{f(r)}\mathscr{E}+\mathcal{W}+\mathbb{A}^{f(r)+1}\mathbb{Z}^n\\ &= \mathbb{A}^{f(r)}(\mathscr{E}+\mathbb{A}\mathbb{Z}^n)+\mathcal{W} = \mathbb{A}^{f(r)}\mathbb{Z}^n+\mathcal{W} = \mathcal{J}.\end{aligned}$$

We generalize the notion of weak product form in [5] to $\mathbb{R}^n$.

Definition 2.1 A digit set $\mathscr{D}\subseteq\mathbb{Z}^n$ is called a weak product-form digit set if there is a product-form digit set $\mathscr{D}'$ with

$$\mathscr{D}' = \mathbb{A}^{f(1)}(\mathscr{E}_1)+\mathbb{A}^{f(2)}(\mathscr{E}_2)+\cdots+\mathbb{A}^{f(r)}(\mathscr{E}_r),$$

such that $\mathscr{D}\equiv\mathscr{D}'\big(\mathrm{mod}\ \mathbb{A}^{f(r)+1}\big)$.

Definition 2.2 We say a set $\mathcal{J}\subseteq\mathbb{Z}^n$ has positive density in $\mathbb{Z}^n$, if

$$\overline{\lim}_{m\to\infty}\frac{1}{(2m)^n}\sharp\big\{\,t\in\mathcal{J}:|t|\le m\big\}>0,$$

where for $t=(t_1,t_2,\cdots,t_n)$, we define $|t|_\infty=\max\{t_1,t_2,\cdots,t_n\}$.

Lemma 2.2 Let $\mathscr{D}\subseteq\mathbb{Z}^n$ be a digit set. Suppose there is a set $\mathcal{J}\subseteq\mathbb{Z}^n$ such that

(i) $\mathcal{J}\subseteq\mathbb{A}\mathcal{J}+\mathscr{D}$;

(ii) $\mathcal{J}$ has positive density in $\mathbb{Z}^n$.

Then $\mathscr{D}$ is a tile digit set.

Proof It is well known that $\mathscr{D}$ is a tile digit set if and only if $\sharp(\mathscr{D}_{\mathbb{A},k})=b^k$. Hence if $\mathscr{D}$ is not a tile digit set, there exists a $k\ge1$ such that $\sharp(\mathscr{D}_{\mathbb{A},k})<b^k$. From

$$\begin{aligned}\mathscr{D}_{\mathbb{A},km} &= \Big\{\sum_{j=0}^{km-1}\mathbb{A}^j d_j : d_j\in\mathscr{D}\Big\}\\ &= \mathbb{A}^{(m-1)k}\mathscr{D}_{\mathbb{A},k}+\cdots+\mathbb{A}\mathscr{D}_{\mathbb{A},k}+\mathscr{D}_{\mathbb{A},k} = (\mathscr{D}_{\mathbb{A},k})_m,\end{aligned}$$

we deduce that $\sharp\mathscr{D}_{\mathbb{A},km}\le(\sharp\mathscr{D}_k)^m$. Therefore

$$\lim_{m\to\infty}\frac{\sharp\mathscr{D}_{\mathbb{A},km}}{b^{km}}\le\lim_{n\to\infty}\frac{(\sharp\mathscr{D}_{\mathbb{A},k})^m}{b^{km}}=\lim_{m\to\infty}\Big(\frac{\sharp\mathscr{D}_{\mathbb{A},k}}{b^k}\Big)^m\le\lim_{m\to\infty}\Big(\frac{b^k-1}{b^k}\Big)^m=0,$$

$$\frac{\sharp\mathscr{D}_{\mathbb{A},m+1}}{b^{m+1}}=\frac{\sharp(\mathscr{D}_{\mathbb{A},m}+\mathscr{D})}{b^{m+1}}\le\frac{\sharp\mathscr{D}_{\mathbb{A},m}}{b^m}\cdot\frac{\sharp\mathscr{D}}{b}=\frac{\sharp\mathscr{D}_{\mathbb{A},m}}{b^m},$$

which means that $\frac{\sharp\mathscr{D}_m}{b^m}$ is non-increasing on n. This yields that

$$\lim_{m\to\infty}\frac{\sharp\mathscr{D}_m}{b^m}=0.$$

Now, if we repeat the inclusion in (i) for k times, we have

$$\mathcal{J}\subseteq\mathbb{A}^k\mathcal{J}+\mathscr{D}_k\subseteq\mathbb{A}^k\mathbb{Z}^n+\mathscr{D}_k.$$

So the density of $\mathcal{J}$ is not larger than the density of $\mathbb{A}^k\mathbb{Z}^n + \mathscr{D}_k$, which is at most $\sharp\frac{\mathscr{D}_k}{b^k}$. We see from this that the density of $\mathcal{J}$ is 0. This is a contradiction; so $\mathscr{D}$ must be a tile digit set.

Theorem 2.3 If $\mathscr{D}$ is a weak product-form digit set, then $\mathscr{D}$ is a tile digit set.

Proof Let $\mathscr{D}'$ be the associated product-form digit set an in the definition and $\mathcal{J} = \mathcal{W} + \mathbb{A}^{f(r)}\mathbb{Z}^n$ be the translation set as in Theorem 2.2. Clearly $\mathcal{J}$ has positive density in $\mathbb{Z}^n$. On the other side, there is a $t \in \mathbb{Z}^n$ such that $\mathscr{D} = \mathscr{D}' + (\mathbb{A}^{f(r)+1})t$, so

$$\begin{aligned}\mathbb{A}\mathcal{J} + \mathscr{D} &= \mathbb{A}\mathcal{W} + \mathbb{A}^{f(r)+1}\mathbb{Z}^n + \mathscr{D}' + \mathbb{A}^{f(r)+1}t \\ &= \mathbb{A}^{f(r)}\mathscr{E} + \mathcal{W} + \mathbb{A}^{f(r)+1}\mathbb{Z}^n + \mathbb{A}^{f(r)+1}t \\ &= \mathbb{A}^{f(r)}(\mathscr{E} + \mathbb{A}\mathbb{Z}^n + \mathbb{A}t) + \mathcal{W} = \mathbb{A}^{f(r)}\mathbb{Z}^n + \mathcal{W} = \mathcal{J}.\end{aligned}$$

So $\mathcal{J}$ satisfies the two conditions of Lemma 2.2. Therefore $\mathscr{D}$ is a tile digit set.

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整数自仿Tiling的Tile数字集

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摘 要: 关于由一个扩张矩阵 $\mathbb{A} \in M_n(\mathbb{Z})$ 和数字集 $\mathscr{D} = \{d_1, d_2, \cdots, d_m\} \subseteq \mathbb{Z}^n$ 生成的整数自仿Tiling，已经有很多研究结果。其中一个重要的问题是判定一个数字集在什么条件下能生成一个Tile。在一维情况下，已知结果有：标准数字集，乘积形式数字集，弱乘积形式数字集都是Tile数字集。在本文中，我们把弱乘积形式的概念推广到高维，并证明它们都是Tile数字集。

关键词: 整数自仿Tile；标准数字集；乘积形式数字集；弱乘积形式数字集